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Generalized Exponential Operators in the Continuation of the Confluent Hypergeometric Functions

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Use of successive continuation procedures for the confluent hypergeometric functions is developed, analyzed, and then applied to the evaluation of Bessel functions $K_0(x)$, $K_1(x)$, $K_{i\nu}(x)$, as well as Fourier series $\sum_{n=1}^{\infty} \cos(nc) K_0(nb)$. A Miller formula for $K_{i\nu}(x)$ and $K'_{i\nu}(x)$ is also discussed.

1. INTRODUCTION

The need to evaluate various functions for a succession of arguments and fixed parameters has motivated us to consider generalized two-dimensional exponential series

$$I + hT_1 + \frac{h^2}{2!} T_2 T_1 + \frac{h^3}{3!} T_3 T_2 T_1 + \cdots \tag{1}$$

with $T_1, T_2, T_3,...$ as a sequence of operators. These series generalize the matrix exponential series $\exp(T) = I + hT + \cdots$, which is used to continue solutions of homogeneous differential equations with constant coefficients, whereas we will use our generalized series to continue the confluent hypergeometric functions $U(a, b, x)$ and $M(a, b, x)$, which satisfy linear homogeneous equations whose coefficients are not constant. In this computational approach one needs to start out with well-defined function values. In [14] we gave a general formula for the confluent functions $U(a, b, x)$, but it is still of interest to look at some special cases.

In order to demonstrate the utility of the method we shall discuss the computation of the Bessel functions $K_0(x)$, $K_1(x)$ and $K_{i\nu}(x)$. One has a useful scientific application of our method with regard to the computation of Fourier series of the form

$$\sum_{n=1}^{\infty} \cos(nc) K_0(nb), \tag{2}$$

where $b > 0$ is small. These series were discussed in the recent paper of Fripiat and

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Delhalle [4] in connection with the calculation of Madelung energies for polymers. On the other hand, the Bessel function of pure imaginary parameter

$$K_{i\nu}(x) = \int_0^\infty \exp(-x \cosh t) \cos(\nu t) \, dt$$

as well as its derivative are important functions which have many occurrences. Some effective integration procedures were developed by Boris and Oran [2] in support of calculations of molecular cross-sections. These functions also occur in connection with *bremstrahlung* in Jackson [5]. Their occurrence in number theory is described by Cartier [3]. A rare treatment of these functions from the point of view of special functions is to be found in Lebedev [7], but the usual formulas have remained entirely inadequate for the determination of their values.

Quite recently in [13] we reported on a continuation operator method for evaluating incomplete Γ-functions, a method which was related to, but a little distinct from, the Taylor series method considered by Rudenberg [9]. The approach considered here is more general than that in [13], but does not subsume our earlier paper.

2. The First Continuation Operator

The confluent hypergeometric function $U(a, b, x)$ and modified confluent hypergeometric function $F(a, b, x) = [\Gamma(b-a)/\Gamma(b)]\, M(a, b, x)$ have the integral representations

(3) $$\Gamma(a)\, U(a, b, x) = e^x \int_1^\infty e^{-xt}(t-1)^{a-1} t^{b-a-1} \, dt, \tag{3}$$

where $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(x) > 0$, and

$$\Gamma(a)\, F(a, b, x) = \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1} \, dt, \tag{4}$$

where $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(b-a) > 0$.

These functions satisfy identical recursion relations

$$(b-a)\, U(a, b, x) - (b+x)\, U(a, b+1, x) + xU(a, b+2, x) = 0, \tag{5}$$

$$(b-a)\, F(a, b, x) - (b+x)\, F(a, b+1), x) + xF(a, b+2, x) = 0. \tag{6}$$

It is easy to derive the Taylor expansions

$$U(a, b, x-h) = e^{-h}(U(a, b, x) + hU(a, b+1, x) + h^2 U(a, b+2, x)/2! + \cdots), \tag{7}$$

$$F(a, b, x-h) = e^{-h}(F(a, b, x) + hF(a, b+1, x) + h^2 F(a, b+2, x)/2! + \cdots), \tag{8}$$

and thus we can introduce the operator series

$$T_{x,y} = e^{-h} \left(I + hT_1 + \frac{h^2}{2!} T_2 T_1 + \frac{h^3}{3!} T_3 T_2 T_1 + \cdots \right), \tag{9}$$

constructed from transformations on two-vectors defined by

$$T_n(u, v) = \left(v, \left(\frac{a - b - n + 1}{x} \right) u + \left(\frac{b - 1 + x + n}{x} \right) v \right). \tag{10}$$

One now has the formal continuation properties

$$T_{x,y}(U(a, b, x), U(a, b + 1, x)) = (U(a, b, y), U(a, b + 1, y)), \tag{11}$$

$$T_{x,y}(F(a, b, x), F(a, b + 1, x)) = (F(a, b, y), F(a, b + 1, y)). \tag{12}$$

The modified Wronskian for the two pairs of functions is

$$W(x) = \det \begin{pmatrix} U(a, b, x) & U(a, b + 1, x) \\ F(a, b, x) & F(a, b + 1, x) \end{pmatrix} = \frac{\Gamma(b - a)}{\Gamma(a)} e^x x^{-b}. \tag{13}$$

Thus if a is not a negative integer then $T_{x,y}$ continues two independent pairs of functions.

LEMMA. *If $\beta = \mathrm{Re}(b) > 1$ then one has the estimate*

$$|\Gamma(a)\, U(a, b, x)| \leqslant |e^x|\, |x|^{1-\beta} \Gamma(\beta - 1). \tag{14}$$

Proof. It is easy to see from the integral representation (3) that (14) holds when $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(x) > 0$. The condition on $\mathrm{Re}(a)$ is eliminated by considering the confluent recursive relations in the first index. The condition on x is eliminated by considering successive continuations as well as the estimate

$$\begin{aligned} |U(a, b, y)| &\leqslant |e^h| \, (|U(a, b, x)| + |h|\, |U(a, b + 1, x)| \\ &\qquad + |h|^2 \, |U(a, b + 2, x)|/2! + \cdots) \\ &\leqslant |e^y|\, |y|^{1-\beta} \Gamma(\beta - 1), \end{aligned}$$

which is a valid estimate when $h = x - y$ satisfies $|h/x| < 1$.

THEOREM. *If a is not a negative integer then the series for $T_{x,y}(u, v)$ converges for arbitrary (u, v) provided $|h/x| < 1$ where $h = x - y$.*

Proof. One sees this from the observation that (8) converges without restriction and that (7) converges if $|h/x| < 1$. Thus $T_{x,y}(u, v)$ converges for two linearly independent choices of (u, v).

3. Continuation through the Differential Equation

The functions $M(a, b, x)$ and $U(a, b, x)$ satisfy the differential equation

$$x \frac{d^2 w}{dx^2} + (b - x)\frac{dw}{dx} - aw = 0. \tag{15}$$

If D denotes d/dx then the Leibnitz formula $D^n(uv) = (D^n u)v + \binom{n}{1}(D^{n-1}u)\,Dv + \cdots + uD^n v$ leads to the relation $xd^{n+2}w/dx^{n+2} + (b+n-x)\,d^{n+1}w/dx^{n+1} - (a+n)\,d^n w/dx^n = 0$. If one now considers the Taylor expansion pair

$$\begin{aligned} w(x+h) &= w(x) + h(dw/dx) + h^2(d^2w/dx^2)/2! + \cdots, \\ \left.\frac{dw}{dx}\right|_{x+h} &= dw/dx + h(d^2w/dx^2) + h^2(d^3w/dx^3)/3! + \cdots, \end{aligned} \tag{16}$$

then one is motivated to define the continuation operator

$$S_{x,y} = I - hS_1 + \frac{h^2}{2!} S_2 S_1 - \frac{h^3}{3!} S_3 S_2 S_1 \pm \cdots, \tag{17}$$

where $h = x - y$ and where

$$S_n(u, v) = \left(v, \left(\frac{n-1+a}{x}\right) u + \left(\frac{x-b-n+1}{x}\right) v\right). \tag{18}$$

THEOREM. *If a is not a negative integer then the series for the continuation operator $S_{x,y}$ converges whenever $|h/x| < 1$, where $h = x - y$.*

The relations between $T_{x,y}$ and $S_{x,y}$. One has the relation $DU(a, b, x) = U(a, b, x) - U(a, b+1, x)$ and the relation $DM(a, b, x) = M(a, b, x) - ((b-a)/b)\,M(a, b+1, x)$. If one defines $W(u, v) = (u, u - v)$ then one obtains

$$T_{x,y} = WS_{x,y}W. \tag{19}$$

We will subsequently show tat $T_{x,y}$ and $S_{x,y}$ have different numerical behavior. If one introduces $R_n = (-h/n)S_n$ then one has

$$R_n(u, v) = (-vh/n, -((n-1+a)u + (x-b-n+1)v)h/xn) \tag{20}$$

then

$$S_{x,y} = I + R_1 + R_2 R_1 + R_3 R_2 R_1 + \cdots. \tag{21}$$

By virtue of the update relations $(n+1)/h = n/h + 1/h$ and $x(n+1)/h = xn/h + x/h$ one can evaluate R_n with two multiplications and two divisions. It should be clear that in computation with small x one might modify the update method to avoid division by small quantities.

The transitivity relations

$$S_{x,z} = S_{y,z} S_{x,y}, \qquad T_{x,z} = T_{y,z} T_{x,y}, \tag{22}$$

allow one to define $T_{x,y}$ and $S_{x,y}$ uniquely for all positive numbers x and y.

4. Stability of the Continuation Operators

The esssential stability of the continuation $T_{x,y}((U(a,b,x),\ U(a,b+1,x)) = (U(a,b,y), U(a,b+1,y))$ (with $y \leqslant x$) comes about because $T_{x,y}$ decreases the magnitude of $(F(a,b,x), F(a,b+1,x))$. Similarly one has the essential stability of $T_{x,y}((F(a,b,x), F(a,b+1,x)) = (F(a,b,y),\ F(a,b+1,y))$ (with $y \geqslant x$). The term "essential stability" means that the continuation passes on rounding errors and is not error correcting, unlike the case considered in [13]. Nevertheless the process is dependent on an error-cancallation phenomenon because the extraneous solutions cancels out when one proceeds in the correct direction.

It is useful to note that the coefficients in the series $F(a,b,y) = e^{-h}(F(a,b,x) + hF(a,b+1,x) + h^2F(a,b+1,x)/2! + \cdots)$ are related by a continued fraction. Indeed

$$\frac{F(a,b+n+1,x)}{F(a,b+n,x)} = \cfrac{b+n-a}{(b+n+x) - x\dfrac{F(a,b+n+2,x)}{F(a,b+n+1,x)}} \tag{23}$$

leads to the continued fraction

$$\frac{F(a,b+1,x)}{F(a,b,x)} = \frac{(b-a)|}{|(b+x)} - \frac{x(b+1-a)|}{|(b+1+x)} - \frac{x(b+2-a)|}{|(b+2+x)} - \cdots. \tag{23}$$

The structure of the above continued fraction is simpler in the economy of operations than the related one for $M(a,b+1,x)/M(a,b,x)$ discussed by Perron [8, p. 126]. Simple linear algebra applied to (5) and (6) allows one to derive the formula

$$\begin{aligned}&\frac{(b-a)|}{|(b+x)} - \frac{x(b+1-a)|}{|(b+1+x)} - \cdots - \frac{x(b+n-a)|}{|(b+n+x)} \\ &\quad = \frac{F(a,b+1,x)}{F(a,b,x)}\left(1 - \frac{F(a,b+n+1,x)}{U(a,b+n+2,x)}\,\frac{U(a,b+1,x)}{F(a,b+1,x)}\right) \\ &\quad\times\left(1 - \frac{F(a,b+n+1,x)}{U(a,b+n+2,x)}\,\frac{U(a,b,x)}{F(a,b,x)}\right)^{-1}.\end{aligned} \tag{25}$$

If $\text{Re}(a) > 0$ and $\text{Re}(x) > 0$ then one sees directly from (3) and (4) that $\lim_{n\to\infty} F(a, b+n+1, x)/U(a, b+n+2, x) = 0$. This is thus the classic argument for the convergence of continued fractions generated by three term recurrence relations.

Thus the real nature of the continuation operators as opposed to the ideal Taylor series is not straightforward. For this reason it surprised us that it was possible to demonstrate that $T_{x,y}$ is error correcting at the quotient level and that one can actually make assertions about the results entirely in terms of observable quantities, the initial and final values computed and that the second solution is not directly present in the stability formulas.

In [13] we considered a function T and wrote $T(u(1+\varepsilon)) = T(u)(1+\theta(u, \varepsilon)\varepsilon)$ to motivate the definition of $\theta_T(u) = uT'(u)/T(u)$, which was termed the *stability factor* of T at u. If T is a two-dimensional transformation of the form $T(u, v) = (T_1(u, v), T_2(u, v))$, where each component is a complex function of two complex variables, then one defines the *stability matrix*

$$\theta_T(u, v) = \begin{bmatrix} \dfrac{u}{T_1(u,v)} \dfrac{\partial T_1}{\partial u}(u, v) & \dfrac{v}{T_1(u,v)} \dfrac{\partial T_1}{\partial v}(u, v) \\ \dfrac{u}{T_2(u,v)} \dfrac{\partial T_2}{\partial u}(u, v) & \dfrac{v}{T_2(u,v)} \dfrac{\partial T_2}{\partial v}(u, v) \end{bmatrix}. \tag{26}$$

The generalization to any number of dimensions is obvious.

If S and T are dimensionally compatible transformations for which one can form the composition TS then one has the chain rule

$$\theta_{TS}(\mathbf{w}) = \theta_T(S(\mathbf{w}))\, \theta_S(\mathbf{w}) \tag{27}$$

for a vector $\mathbf{w}$, a rule which is valid for differentiable functions which do not vanish at the arguments.

Given a transformation of the form $T(u, v) = (Au + Bv, Cu + Dv)$ we shall define the *associated fractional linear map* by

$$H(w) = F(u, v) = \frac{Au + Bv}{Cu + Dv}, \tag{28}$$

where $w = u/v$. If $P(u, v) = u$ and $Q(u, v) = v$ then it is easy to demonstrate the theorem to follow.

THEOREM. *If T, F, and H are as before and $w = u/v$ then*

$$\theta_F(u, v) = (\theta_H(w), \theta_H(w)) \tag{29}$$

and

$$\theta_H(w) = \frac{uv}{PT(u, v)\, QT(u, v)} \det T = \det \theta_T(u, v). \tag{30}$$

COROLLARY 1. *Let T be either of the continuation operators $T_{x,y}$ or $S_{x,y}$. One has the formula*

$$\det \theta_T(u, v) = \frac{uv}{PT(u, v)\, QT(u, v)} e^{-h}(y/x)^{-b}, \tag{31}$$

where $h = x - y$.

Proof. One notes that $\det T = W(y)/W(x) = e^{-h}(y/x)^{-b}$, where $W(x)$ is the Wronskian (13).

COROLLARY 2 (Stability at quotient level). *If $a > 0$, and $x > y > 0$, then one has*

$$\begin{aligned} 0 &\leqslant \det \theta_S(U(a, b, x), DU(a, b, x)) < 1, \\ 0 &\leqslant \det \theta_T(U(a, b, x), U(a, b+1, x)) < 1, \end{aligned} \tag{32}$$

where $S = S_{x,y}$ and $T = T_{x,y}$.

Proof. We demonstrate the proof for S. One obtains

$$\det \theta_S(U(a, b, x), DU(a, b, x)) = \frac{U(a, b, x)\, DU(a, b, x)}{U(a, b, y)\, DU(a, b, y)} e^{-h}(y/x)^{-b} \tag{33}$$

from (31). One obtains $U(a, b, x)/U(a, b, y) < 1$ directly from the integral representation (3). When $h = x - y$ one also sees that

$$\begin{aligned} &(DU(a, b, x)/DU(a, b, y))\, e^{-h}(y/x)^{-b} \\ &= \int_h^\infty e^{-t}(t-h)^a (y+t)^{b-a-1}\, dt \Big/ \int_0^\infty e^{-t} t^a (y+t)^{b-a-1}\, dt < 1. \end{aligned} \tag{34}$$

5. APPLICATION TO THE EVALUATION OF $K_0(x)$ AND $K_1(x)$.

The Bessel functions $K_0(x)$ and $K_1(x)$ are special cases of Macdonald's functions

$$K_\nu(x) = \int_0^\infty e^{-x \cosh u} \cosh \nu u\, du. \tag{35}$$

The relation $U(\nu + 1/2, 2\nu + 1, 2x) = \pi^{-1/2} e^x (2x)^{-\nu} K_\nu(x)$ yields

$$\begin{aligned} \pi^{1/2} U(1/2, 1, 2x) &= e^x K_0(x), \\ \pi^{1/2} DU(1/2, 1, 2x) &= e^x (K_0(x) - K_1(x))/2. \end{aligned} \tag{36}$$

If $S_{x,y}$ denotes the continuation operator (17) for $(U(1/2, 1, x), DU(1/2, 1, x))$ then

$$S_{2x,2y}(e^x K_0(x), e^x(K_0(x) - K_1(x))/2) = (e^y K_0(y), e^y(K_0(y) - K_1(y))/2). \tag{37}$$

TABLE I

Degree of Precision That Was Obtained in Downward Continuation and the Number of Terms That Were Summed to Obtain Convergence

x	$\exp(x)\,K_0(x)$	$\exp(x)\,K_1(x)$	n	c
10	0.39163 19344	0.41076 65704	(initial)	
9	0.41229 5549	0.43462 5245	11	1/10
8	0.43662 3018	0.46314 9092	11	1/9
7	0.46584 5096	0.49807 1575	12	1/8
6	0.50186 3130	0.54217 5910	12	1/7
5	0.54780 7564	0.60027 3858	13	1/6
4	0.60929 7669	0.68157 5945	14	1/5
3	0.69776 1597	0.80656 3479	16	1/4
2	0.84156 8214	1.03347 6846	20	1/3
1	1.14446 3078	1.63615 3482	28	1/2
0.5	1.52410 9383	2.73100 9694	28	1/2
0.4	1.66268 2086	3.25867 3862	13	1/5
0.3	1.85262 7296	4.12515 7740	15	1/4

In the next section we shall explain why this continuation procedure behaves even better than one might expect. The number of terms required to obtain a given accuracy does not vary much with x and is essentially a function of $c = h/x$ ($h = x - y$). Table I shows the results obtained on a HP calculator with floating-point accuracy of 10 digits. The series summation was terminated when there was no detectable change in the partial sum, whereupon the content of the term counter register was also noted.

6. Application to the Evaluation of Madelung Sums

Te series we seek to evaluate are of the form

$$M = \sum_{n=1}^{\infty} \cos(nc)\,K_0(nb) = \pi^{1/2} \sum_{n=1}^{\infty} \cos(nc)\,e^{-nb}\,U(2nb), \tag{38}$$

where $U(x) = U(1/2, 1, x)$ is as in (36). Despite the exponential convergence of the series, the computation was observed to be excessive when b is small. Moreover, in the polymer calculations it is seen that the values of c are determined by sporadic intermolecular distances and the usual methods for evaluating discrete Fourier transforms are not appropriate.

Commencing with easily determined initial data we updated the confluent functions with

$$S_{2nb,2(n-1)b}(U(2nb), U'(2nb)) = (U(2(n-1b), U'(2(n-1)b)) \tag{39}$$

for $n = N, N-1, \ldots, 2$. When b is small one commences with large n and the continuation series converges at the same rate as the geometric series $1/n + 1/n^2 + 1/n^3 + \cdots$. Thus most function values $U(2nb)$ are generated very rapidly. The quantities $\exp(-nb)\cos(nc)$ are obtained through backward recursion using the complex functional relation $\exp(-n(b+ic)) = \exp(-(n+1)(b+ic))$ $\exp(b+ic)$, but this recursion is vulnerable because initial values at large arguments have a large intrinsic error and because the argument drifts during the recursion.

When $x > y$ the continuation operator $S_{x,y}$ is stable and the behavior of the process under iteration is quite good. An important contributor to this good behavior is that series (17) becomes a series of positive terms because the signs of the derivatives of $U(a, b, x)$ alternate. Moreover, with proper programming, there is no argument drift. Additional good behavior comes about because $S_{x,y}$ is strongly error correcting at the quotient level. In order for errors to be passed in the result $S_{x,y}(u, v) = (u', v')$ both values u' and v' have to be perturbed by the same scale factor.

Our recursive economies have allowed us to experiment with Madelung sums with parameter values around $b = 10^{-2}$, for which we were able to obtain satisfactory results. Nevertheless, for very small parameter values the method in [4] is attractive.

7. Bessel Functions of Pure Imaginary Parameter

The Bessel functions of pure imaginary parameter $K_{i\nu}(x)$ are real functions and are discussed in Lebedev [7]. The usual formulas for $K_{i\nu}(x)$ lead to severe cancellation-error difficulties. In physical applications (Jackson [6]) it is also desirable to compute the derivatives $K'_{i\nu}(x)$.

TABLE II

Function Values and Starting Indices for the Miller Formula (43) for Computing to the Digits Indicated[a]

x	$e^xK_{30i}(x)$	$n(x)$	x	$e^xK_{30i}(x)$	$n(x)$
1	−2.4970483798(−21)	1205	16	1.2497911866(−15)	90
2	−6.4965534241(−21)	640	18	−3.5137264538(−14)	87
3	−2.7189381187(−20)	410	20	7.4161663636(−13)	76
4	8.1506023622(−20)	319	30	1.6539115409(−8)	58
5	2.3408091735(−19)	248	40	1.6332522588(−6)	46
6	4.0109112407(−19)	215	50	1.7921292569(−6)	42
7	−1.5434276621(−18)	183			
8	1.6367027486(−18)	161	60	8.0874542215(−5)	39
9	−1.3472488255(−18)	148	80	4.8890403838(−4)	32
10	8.6847879814(−16)	136	100	1.3748218952(−3)	25
12	2.6398713250(−16)	121	150	5.0906540616(−3)	23
14	−8.0221524854(−15)	103	200	9.3481184525(−3)	19

[a] The last digits shown are not completely certain.

When one applies the continuation operator $T_{x,y}$ to $e^x K_{i\nu}(x)$, interpreted as a confluent function, one experiences success, but $S_{x,y}$ turns out to be unsuitable. Examination of Table II shows that $e^x K_{i\nu}(x)$ decreases rapidly as x decreases. With $T_{x,y}$ this decrease is brought about by multiplication by $\exp(-(x-y))$, but with $S_{x,y}$ this decrease is brought about by subtraction and thus leads to poor number representation. For best results $T_{x,y}$ needs to be applied successively with the constraint $(h/x) < 1/\nu$ with $h = x - y > 0$ and $\nu > 0$.

The continuation requires good initial values, but since there are no tables, these are hard to come by. It is for this reason that we now discuss some formulas for generating initial values. It turns out that with our Miller formula approach [14] both $K_{i\nu}(x)$ and $K'_{i\nu}(x)$ can be computed with real arithmetic. Set $U(i\nu + n + \frac{1}{2}, 2i\nu + 1, 2x) = U(n)$. One has

$$K'_{i\nu}(x) = K_{i\nu}(x)\left(\frac{(1+4\nu^2)}{4x}\frac{U(1)}{U(0)} - \frac{1}{2x} - 1\right) \tag{40}$$

$$e^x K_{i\nu}(x) = \left(\frac{\pi}{2x}\right)^{1/2}\left(1 + \frac{(1^2+4\nu^2)}{4^1\cdot 1!}\frac{U(1)}{U(0)} + \frac{(1^2+4\nu^2)(3^2+4\nu^2)}{4^2\cdot 2!}\frac{U(1)}{U(0)}\frac{U(2)}{U(1)} + \cdots\right)^{-1}. \tag{41}$$

The quotients $U(n)/U(n-1)$ are real and have the expansions

$$\frac{U(n)}{U(n-1)} = \frac{4|}{|2(n+x)} - \frac{(2n+1)^2+4\nu^2|}{|2(n+1+x)} - \frac{(2n+3)^2+4\nu^2|}{|2(n+2+x)} - \cdots. \tag{42}$$

Backward recursion with the relation

$$\frac{U(n)}{U(n-1)} = \frac{4}{8(n+x) - ((2n+1)^2+4\nu^2)\,U(n+1)/U(n)} \tag{43}$$

along with nested operations was demonstrated in [14] to be an attractive approach to the evaluation of (41). An ALGOL code is available on request. We have observed that the starting index $n(\nu, x)$ for evaluating our Miller formula to fixed number of significant digits satisfies the relations $n(\nu, x) \leqslant n(\nu', x)$ for $0 < \nu \leqslant \nu'$ and $n(\nu, x) \geqslant n(\nu, x')$ for $0 < x \leqslant x'$. An approximation to the starting index function in Table II is thus good for producing values of $K_{i\nu}(x)$ in the region $1 \leqslant x \leqslant \infty$ and $0 \leqslant \nu \leqslant 30$.

Another effective algorithm for computing the Bessel functions $K_{i\nu}(x)$ is described by Boris and Oran [2]. This paper also refers to more extensive tables of these functions.

Thacher [15] used an entirely different approach to develop a formula from which one can radily deduce (40). Our Miller formula (41) is closely related to the one devised by Temme [11, 12].

We found the operator method $T_{x,y}$ to be successful but found it annoying that the recursive quantities generated need to be complex numbers.

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